ISRAEL JOURNAL OF MATHEMATICS 141 (2004), 185–210

AFFINE PROPERTIES AND INJECTIVITY OF QUASI-ISOMETRIES

BY

I. A. VESTFRID

Department of Mathematics, Technion — Israel Institute of Technology
32000 Haifa, Israel
e-mail: vestig@tx.technion.ac.il

ABSTRACT

We approximate ε-quasi-isometries between finite-dimensional Banach spaces by linear near-isometries. In this way we improve and extend a theorem of John. We also improve results of Gevirtz on injectivity criteria for quasi-isometries. Our approach is to show that ε-quasi-isometries almost satisfy the Jensen functional equation and to use then known facts about linear approximation of approximate solutions of Jensen's equation.

1. Introduction

The classical Mazur–Ulam theorem [MU] asserts that a surjective isometry between real normed spaces is affine. Moreover, John [J3] showed that any local isometry which maps an open connected subset of a real normed space X onto an open subset of another real normed space Y is the restriction of an affine isometry of X onto Y. The proofs are based on showing that such maps satisfy the Jensen functional equation $2f(\frac{x+y}{2}) - f(x) - f(y) = 0$ (in John's theorem the equation is satisfied locally), and the continuity then implies that they are actually affine. The example of the function $t \mapsto (t, |t|)$ from $\mathbf{R}$ to l_∞^2 shows that the hypothesis that the isometry is a local homeomorphism cannot be omitted. Note also that these conclusions are not valid for complex normed spaces (just consider complex conjugation on $\mathbf{C}$).

In this article we study the approximation of quasi-isometries by affine maps as well as some of their other geometric properties. Throughout, X and Y are real Banach spaces.

Received April 16, 2001 and in revised form April 7, 2003

Definitions: Let $\varepsilon \geq 0$.

1) If $\Omega \subset X$ and $f: \Omega \to Y$, then at each point x of Ω one defines $D^+ f(x)$ and $D^- f(x)$ as the upper and lower limits, respectively, of $\|f(x) - f(y)\|/\|x - y\|$ as y tends to x. Following John [J2], a map of an open subset Ω of X into Y is said to be (m, M)**-quasi-isometric** if it satisfies the following two conditions:

(1) f is a local homeomorphism, i.e., every point $x \in \Omega$ has an open neighborhood V such that f is a homeomorphism of V onto an open subset of Y;

(2) $0 < m \leq D^- f(x)$ and $D^+ f(x) \leq M$ for every $x \in \Omega$.

f is called an ε**-quasi-isometry** if it is an (m, M)-quasi-isometry satisfying $(1 + \varepsilon)^{-1} \leq m \leq M \leq (1 + \varepsilon)$, and simply a **quasi-isometry** if it is an ε-quasi-isometry for some $\varepsilon \geq 0$.

2) A map f from a subset S of X into Y is called ε**-rigid** if

$$(1 + \varepsilon)^{-1} \|y - x\| \leq \|f(y) - f(x)\| \leq (1 + \varepsilon) \|y - x\|$$

for all $x, y \in S$.

Note that if $\dim X = \dim Y < \infty$, then, by the invariance of domains, any ε-rigid mapping of an open set is a local homeomorphism, hence ε-quasi-isometric. (For geometric properties of ε-rigid maps when $\dim X < \dim Y$, see [V1].)

In 1961, John [J1] proved the following local stability theorem for the case when X and Y are the same Euclidean space.

THEOREM 1.1: *Let Ω be a convex domain in l_2^n which contains a ball $B(z, r)$ and is contained in a concentric ball $B(z, R)$. If f is an ε-quasi-isometry in Ω, then there is a linear isometry γ such that*

$$\|f(x) - f(z) - \gamma(x - z)\| \leq k n^{3/2} \varepsilon R^2 / r \quad \text{for every } x \in \Omega,$$

where k is a universal constant.

The following natural question is asked in [BL]: Does Theorem 1.1 hold with an estimate independent of the dimension? Matoušková [M] has answered this question in the negative. She constructed a norm preserving ε-quasi-isometry f of l_2^{2n} onto itself (n is about $\exp\frac{1}{\varepsilon}$) such that the distance of f on the unit ball from any affine mapping of $\mathbf{R}^{2n}$ is at least $1/\sqrt{2}$. On the other hand, the author has shown in 1999 that the estimate in Theorem 1.1 can be replaced by $k\sqrt{n}\varepsilon R^2/r$, where k is a universal constant. Recently this result also has been improved by Kalton [K] to $k(\log n + 1)\varepsilon R^2/r$ (see also Corollary 3.6(iv) below), and this, by the example of Matouškova, is the best estimate for quasi-isometries between Euclidean spaces. The situation is different for other spaces, and we show in Corollary 3.6(vi) that if l_2^n is replaced, for example, by l_∞^n or by l_∞ in Theorem 1.1, then there are universal constants $k, \varepsilon_0 > 0$ so that whenever $0 < \varepsilon < \varepsilon_0$, then there is an affine isometry U such that

$$\|f(x) - U(x)\| \leq k\varepsilon R^2/r \quad \text{for every } x \in \Omega.$$

Thus in this case, the approximation error does not depend on the dimension.

The main purpose of this work was to answer another question asked in [BL], namely: whether the Euclidean norm can be replaced in Theorem 1.1 by other norms? Since there are spaces with no non-trivial linear isometries, it does not make sense to look for an approximation by affine isometries, and the question is whether it is possible to approximate quasi-isometries satisfying $f(0) = 0$ by linear near-isometries, i.e., by linear invertable operators T such that $\|T\| \cdot \|T^{-1}\|$ is close to one. Here we use another stability, namely, the local stability of Jensen's equation. We show that quasi-isometries between Banach spaces belong to a class of approximate solutions of the Jensen functional equation which we call homogeneously approximately midlinear functions (see [V1] for a study of this class of mappings and, in particular, on their approximation by affine maps); we then use the results of [V1] on the approximation of such functions to generalize John's theorem by obtaining affine approximations for quasi-isometries between any two Banach spaces of the same finite dimension.

Definition ([V]): Let X and Y be normed spaces, and let A be a convex subset of X. Let $\varepsilon \geq 0$. A function $f\colon A \to Y$ is said to be **homogeneously ε-approximately midlinear** if

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leq \varepsilon \|x - y\|$$

for all $x, y \in A$.

We shall abbreviate "approximately midlinear" by AML.

The last section of the article deals with the problem on the injectivity of quasi-isometries.

In Section 2 we prove the Main Lemma (Lemma 2.1), which is an improvement of [G2, Proposition 2] of Gevirtz. It follows, for example, from this lemma and from Lemma 2.5 that an ε-quasi-isometry of a ball $B(z,r)$ is homogeneously $C\varepsilon$-AML on any proper concentric sub-ball $B(z,\rho)$ with C depending only on the ratio r/ρ. The proof uses the technique of Lindenstrauss and Szankowski from [LS].

Combining the Main Lemma with some results from [V1] enables us to obtain in Section 3 affine approximations of quasi-isometries. Then applying known facts on closeness of linear near-isometries to isometries we show that ε-quasi-isometries (with a small ε) of "nice" spaces are also close to affine isometries.

In Section 4 we use Lemma 2.1 to improve results of Gevirtz [G2] on the injectivity of quasi-isometries defined on open convex sets or on uniform domains.

We use standard notation and terminology. As usual (x, y) and $[x, y]$ denote the open and closed straight line segments joining the points x and y. The r-neighborhood of a set A is denoted by $B(A, r)$, and we abbreviate $B(\{z\}, r)$ by $B(z, r)$. For short we also write $B(0, r) = B(r)$ and $B(1) = B$ (or B_E when we need to specify the space). The closure, diameter and convex hull of a set A are denoted by $\bar{A}$, $\operatorname{diam} A$ and $\operatorname{co} A$, respectively. The cardinality of a set A is denoted by $|A|$. The Banach–Mazur distance between normed spaces X and Y is denoted by $d(X, Y)$. If Banach space X has non-trivial type p, we denote by $T_p(X)$ its p-type constant.

2. The Main Lemma

The next lemma shows that locally any quasi-isometry of an open set is homogeneously AML. The assertion is a modification of the Proposition in [LS], and the proof is similar, but we need to be somewhat more careful because our map is not defined on the whole space.

LEMMA 2.1 (Main Lemma): *Let $\varepsilon > 0$, and let $f: B_X(z, r) \to Y$ be ε-quasi-isometric. Then f is homogeneously 6ε-AML on the concentric ball $\bar{B}(z, r/5)$.*

We state as two lemmas some basic facts due to Nevanlinna [N] and John [J3] (see also [BL, Chapter 14]), and we give them without proof.

LEMMA 2.2: *[N] Let A be a convex subset of X, and let $f: A \to Y$ satisfy $D^+ f(x) \le M$ for all $x \in A$. Then $\|f(y) - f(x)\| \le M\|y - x\|$ for all $x, y \in A$.*

LEMMA 2.3: *[J3] Let $B(z, r) \subset X$, and let $f: B(z, r) \to Y$ be (m, M)-quasi-isometric. Then*

(i) *$m\|y - x\| \le \|f(y) - f(x)\| \le M\|y - x\|$ for all $x, y \in B(z, rm/M)$.*

(ii) *$B(f(z), mr) \subseteq f(B(z, r))$.*

Proof of Lemma 2.1: Fix any $x, y \in \bar{B}(z, r/5)$.

If $\varepsilon > 1/11$, then by Lemma 2.2

$$\left\|f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2}\right\| \leq \frac{1}{2}\left(\left\|f\left(\frac{x+y}{2}\right) - f(x)\right\| + \left\|f\left(\frac{x+y}{2}\right) - f(y)\right\|\right)$$
$$\leq (1+\varepsilon)\|x-y\|/2 < 6\varepsilon\|x-y\|.$$

Suppose now that $\varepsilon \leq 1/11$. By translation and scaling, we can assume that $y = -x$, $f(x) = -f(y)$ and $\|x-y\| = 2$. Under this normalization we need to prove that $\|f(0)\| \leq 12\varepsilon$. Note that under the normalization $\|z\| \leq r/5$ and $r \geq 5$, hence $B(4) \subset B(z,r)$. Since $(1+\varepsilon)^2(e^{8/11} + (1+\varepsilon)^2) < 4$, it follows that $B((1+\varepsilon)^2(e^{8/11} + (1+\varepsilon)^2)) \subset B(z,r)$. Hence by Lemma 2.3(i), f is ε-rigid on $B(e^{8/11} + (1+\varepsilon)^2)$. In particular, f is a homeomorphism from $B(e^{8/11} + (1+\varepsilon)^2)$ onto its image. Denote its inverse on $f(B(e^{8/11} + (1+\varepsilon)^2))$ by f^{-1}.

Choose $n \geq 0$ so that $2^{-(n+1)} < 11\varepsilon \leq 2^{-n}$. Then

$$(1+\varepsilon) \leq (1+\varepsilon)^{2^n} \leq (1+\varepsilon)^{1/(11\varepsilon)} < e^{1/11}.$$

Define inductively two sequences $\{w_i\}_{i=0}^{2^{n+3}} \subset X$ and $\{y_i\}_{i=0}^{2^{n+3}+1} \subset Y$ by $w_0 = 0$ and

$$y_{2i} = f(w_{2i}), \qquad y_{2i+1} = -f(w_{2i}),$$
$$w_{2i+1} = f^{-1}(y_{2i+1}), \qquad w_{2i+2} = -f^{-1}(y_{2i+1}).$$

(That $\{y_i\}_{i=0}^{2^{n+3}-1} \subset f(B(e^{8/11} + (1+\varepsilon)^2))$, i.e., that $f^{-1}(y_i)$ is actually well defined, will follow from the inclusion (2.4) below.)

Put for $k = 0, \ldots, 2^{n+2}$,

$$\delta_k = \max\left\{\left\|w_i - w_j\right\| : \left[\frac{i+1}{2}\right] + \left[\frac{j+1}{2}\right] \leq k\right\},$$
$$\Delta_k = \max_{i \leq 2k}\{\|w_i - x\|\}.$$

Note that $\Delta_0 = 1$ and that $\Delta_k = \max_{i \leq 2k}\{\|w_i + x\|\}$ by the symmetry of $\{w_i\}_{i=0}^{2k}$ with respect to the origin. Note also that for $i \leq 2k$,

$$2\|w_i\| = \|w_i - (-w_i)\| \leq \delta_{2k}. \tag{2.1}$$

The following two sublemmas describe the behavior of Δ_k and δ_k.

SUBLEMMA 1: *With the notation as above we have, for $i \leq 2^{n+2}$,*

$$\Delta_i \leq (1+\varepsilon)^{2i}, \tag{2.2}$$
$$y_{2i+1} \in \bar{B}(f(0),\ (1+\varepsilon)^{2i+1} + 1 + \varepsilon). \tag{2.3}$$

Note that by (2.3), Lemma 2.3(ii) and by the choice of n we have, for $i \le 2^{n+2}-1$,

$$y_{2i+1} \in f(\bar{B}((1+\varepsilon)^{2i+2}+(1+\varepsilon)^2)) \subset f(B(e^{8/11}+(1+\varepsilon)^2)). \tag{2.4}$$

Thus, the sequence $\{f^{-1}(y_i)\}_{i=0}^{2^{n+3}-1}$ is well defined.

Proof of Sublemma 1: We shall prove the sublemma by induction on i. Since $f(-x)=-f(x)$ and by Lemma 2.2,

$$\|(-f(0))-f(0)\| \le \|f(x)-f(0)\| + \|f(-x)-f(0)\| \le 2(1+\varepsilon).$$

Thus (2.3) holds for $i=0$; the other claim is trivial for $i=0$. Assuming the sublemma is true for some $i \le 2^{n+2}-1$, we now prove it for $i+1$.

We start with (2.2). It follows from (2.4) for i that

$$y_{2i+1} \in f(B(e^{8/11}+(1+\varepsilon)^2)).$$

Write $\Delta_{i+1} = \|w_j - x\|$ for some $j \le 2i+2$. If $j \le 2i$, then $\Delta_{i+1} = \Delta_i \le (1+\varepsilon)^{2i}$.

If $j = 2i+1$, then $w_j = f^{-1}(y_{2i+1})$ and $f(w_j) = -f(w_{2i})$. Thus

$$\|f(w_j)-f(x)\| = \|f(w_{2i})-f(-x)\|$$

and, by the ε-rigidity of f on $B(e^{8/11}+(1+\varepsilon)^2)$,

$$\begin{aligned}\|w_j - x\| &\le (1+\varepsilon)\|f(w_j)-f(x)\| = (1+\varepsilon)\|f(w_{2i})-f(-x)\| \\ &\le (1+\varepsilon)^2\|w_{2i}+x\|.\end{aligned}$$

If $j = 2i+2$, then $-w_j = f^{-1}(y_{2i+1})$ and $f(-w_j) = -f(w_{2i})$. Thus

$$\|f(-w_j)-f(-x)\| = \|f(w_{2i})-f(x)\|$$

and consequently,

$$\begin{aligned}\|w_j - x\| &\le (1+\varepsilon)\|f(-w_j)-f(-x)\| = (1+\varepsilon)\|f(w_{2i})-f(x)\| \\ &\le (1+\varepsilon)^2\|w_{2i}-x\|.\end{aligned}$$

In both cases $\Delta_{i+1} \le (1+\varepsilon)^2\Delta_i$, and (2.2) holds for $i+1$.

Finally, we prove (2.3) by (2.2) and Lemma 2.2:

$$\begin{aligned}\|y_{2i+3}-f(0)\| &= \|f(w_{2i+2})+f(0)\| \le \|f(x)-f(w_{2i+2})\| + \|f(-x)-f(0)\| \\ &\le (1+\varepsilon)(\|w_{2i+2}-x\|+\|x\|) \le (1+\varepsilon)(\Delta_{i+1}+1) \\ &\le (1+\varepsilon)^{2i+3}+1+\varepsilon. \qquad \blacksquare\end{aligned}$$

SUBLEMMA 2: *With the notation as above*

$$\delta_{2^{m+1}} \geq 2(1-2^m\varepsilon)\delta_{2^m} \quad \textit{for } m=1,\dots,n+1. \tag{2.5}$$

Proof: Fix integers $i \geq j \geq 1$ such that

$$\left[\frac{i+1}{2}\right] + \left[\frac{j+1}{2}\right] \leq 2^{n+2}.$$

We shall show that there are nonnegative integers p, q so that

$$\left[\frac{p+1}{2}\right] \leq \left[\frac{i+1}{2}\right] + 1, \quad \left[\frac{q+1}{2}\right] \leq \left[\frac{j+1}{2}\right] - 1$$

(hence, in particular, $\delta_{[\frac{p+1}{2}]+[\frac{q+1}{2}]} \leq \delta_{[\frac{i+1}{2}]+[\frac{j+1}{2}]}$) and

$$\|w_j - w_i\| \leq \|w_q - w_p\| + 2\varepsilon\delta_{[\frac{i+1}{2}]+[\frac{j+1}{2}]}. \tag{2.6}$$

We then deduce the sublemma as follows: Fix $m \leq n+1$ and choose $i \geq j \geq 0$ such that

$$\left[\frac{i+1}{2}\right] + \left[\frac{j+1}{2}\right] \leq 2^m \quad \text{and} \quad \delta_{2^m} = \|w_j - w_i\|.$$

Let s be the number of times we may apply (2.6) (that is, until $q > 0$). Then $s \leq [\frac{j+1}{2}]$ and we find w_p with $[\frac{p+1}{2}] \leq 2^m$ so that

$$\delta_{2^m} = \|w_j - w_i\| \leq \|w_p\| + 2\left[\frac{j+1}{2}\right]\varepsilon\delta_{2^m} \leq \delta_{2^{m+1}}/2 + 2^m\varepsilon\delta_{2^m},$$

where the last inequality follows from (2.1). (Note that $w_0 = 0$ and that $j \leq i$ implies $2[\frac{j+1}{2}] \leq 2^m$.)

To prove (2.6) note that we can write $\|w_j - w_i\|$ as $\|f^{-1}(y_k) - w_l\|$ with an odd $k \leq j$ and $[\frac{l+1}{2}] = [\frac{i+1}{2}]$. Indeed, if j is odd, then $\|w_j - w_i\| = \|f^{-1}(y_j) - w_i\|$. If j is even, then $\|w_j - w_i\| = \|f^{-1}(y_{j-1}) - (-w_i)\|$. Now similarly (with an odd k), if l is even, then $\|f(w_l) - y_k\| = \|(-f(w_l)) - y_{k-1}\| = \|y_{l+1} - f(w_{k-1})\|$, and if l is odd, then $\|f(w_l) - y_k\| = \|y_{l-1} - f(w_{k-1})\|$. Hence, we can write $\|f(w_l) - y_k\|$ as $\|y_p - f(w_q)\|$ with an even $q < j$ (thus $[\frac{q+1}{2}] \leq [\frac{j+1}{2}] - 1$), $[\frac{p+1}{2}] \leq [\frac{i+1}{2}] + 1$ and $[\frac{p+1}{2}] = [\frac{i+1}{2}] + 1$ only if p is odd. Hence $p < 2^{n+3}$, and we have $w_p = f^{-1}(y_p)$. Then by the ε-rigidity of f on $B(e^{8/11} + (1+\varepsilon)^2)$, we have

$$\begin{aligned} \|w_j - w_i\| - \|w_p - w_q\| =& \|f^{-1}(y_k) - w_l\| - \|y_k - f(w_l)\| \\ &+ \|y_p - f(w_q)\| - \|f^{-1}(y_p) - w_q\| \\ \leq& \varepsilon(\|f^{-1}(y_k) - w_l\| + \|f^{-1}(y_p) - w_q\|) \\ \leq& 2\varepsilon\delta_{[\frac{i+1}{2}]+[\frac{j+1}{2}]}. \end{aligned}$$

∎

We return to the proof of Lemma 2.1. We first show by induction that

$$\delta_{2^{n-k+2}} < \left(\frac{11}{18}\right)^k (e^{8/11}+1) \tag{2.7}$$

for each integer $0 \le k \le n+2$.

By the definitions of δ_m and Δ_i and by (2.2), it follows that for all $m \le 2^{n+2}$

$$\delta_m \le \max_{i+j=m} (\Delta_i + \Delta_j) \le (1+\varepsilon)^{2m} + 1.$$

Taking $m = 2^{n+2}$, the choice of n gives that $\delta_{2^{n+2}} < e^{8/11}+1$, i.e., (2.7) holds for $k=0$. Assume it holds for some $k \ge 0$. The choice of n gives $2(1-2^{n-k+1}\varepsilon) \ge 2(1-2/11) = 18/11$. Hence by (2.5) and the induction hypothesis,

$$\delta_{2^{n-k+1}} \le \delta_{2^{n-k+2}}/(2(1-2^{n-k+1}\varepsilon)) < (11/18)^{k+1}(e^{8/11}+1),$$

which completes the proof of (2.7).

Note also that repeated use of (2.5) gives

$$\delta_{2^{n+2}} \ge 2^{n+1}(\delta_2 - 2\varepsilon(\delta_{2^{n+1}} + \cdots + \delta_2)). \tag{2.8}$$

Finally we obtain by (2.1), (2.8) and $\varepsilon \le 1/11$,

$$\begin{aligned}\|f(0)\| &= \frac{1}{2}\|f(0) - f(f^{-1}(-f(0)))\| = \frac{1}{2}\|f(0)-f(w_1)\| \le \frac{1+\varepsilon}{2}\|w_1\| \\ &\le \frac{1+\varepsilon}{4}\delta_2 \le \frac{3}{11}\delta_2 \le \frac{3}{11}(2^{-(n+1)}\delta_{2^{n+2}} + 2\varepsilon(\delta_{2^{n+1}} + \cdots + \delta_2)) < 12\varepsilon,\end{aligned}$$

because the first term in the parentheses is bounded by $11(e^{8/11}+1)\varepsilon$ (by (2.7) and the choice of n) and the second term is bounded by $2\sum_{k=1}^{\infty}(11/18)^k(e^{8/11}+1)\varepsilon$ (by (2.7)). ∎

Remark 2.4: (i) An inspection of the proof above gives also the following: *If $f\colon B_X((x+y)/2, 2\|x-y\|) \longrightarrow Y$ is ε-quasi-isometric, then*

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| < 6\varepsilon\|x-y\|.$$

(ii) The dependence on ε in the estimate of the Main Lemma is linear. The next simple example shows that this is the correct dependence.

Consider the real function f given by

$$f(t) = \begin{cases} (1+\varepsilon)t, & t \ge 0, \\ t/(1+\varepsilon), & t \le 0. \end{cases}$$

Clearly, f is ε-quasi-isometric. On the other hand, for every $\varepsilon > 0$, $t > 0$

$$\left|f(0) - \frac{f(t)+f(-t)}{2}\right| = \frac{2+\varepsilon}{2+2\varepsilon}\varepsilon t.$$

(iii) Gevirtz [G2] was the first to establish that an ε-quasi-isometry of a ball $B(z,r)$ is homogeneously $k(\varepsilon)$-AML on some concentric sub-ball $B(z,\rho)$. It follows from his result that if $\rho \le r/3$, then $k(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$. He used different arguments and gave the estimate $k(\varepsilon) = C\varepsilon^{0.1216\dots}$ where the constant C depends only on the ratio r/ρ.

By scaling, Lemma 2.1 gives the approximate midlinearity on $\bar{B}_X$ of an ε-quasi-isometry defined on $B_X(1+\delta)$ for $\delta \ge 4$. The next statement extends this result for all $\delta > 0$.

LEMMA 2.5: *Let $0 < \delta \le 4$, and let $f\colon B_X(1+\delta) \to Y$ be ε-quasi-isometric. Then f is homogeneously $48\varepsilon/\delta$-AML on $\bar{B}_X$.*

Proof: Fix any $x, y \in \bar{B}_X$.

If $\delta \ge 2\|x-y\|$, then $B(\frac{x+y}{2}, 2\|x-y\|) \subset B(1+\delta)$ and Remark 2.4(i) applies. (Recall that $\delta \le 4$.)

Suppose that $\delta < 2\|x-y\|$. We shall use the following easily checked identity (shown to me by Y. Benyamini):

Let $\{v_i\}_{i=-N}^{N}$ be points in a linear space. Then

$$v_0 - \frac{v_{-N}+v_N}{2} = \sum_{|i|\le N-1} (N-|i|)\Big(v_i - \frac{v_{i-1}+v_{i+1}}{2}\Big). \tag{2.9}$$

Put $N = [2\|x-y\|/\delta] + 1$, and consider the partition $\{w_i\}_{i=-N}^{N}$ of the segment $[x,y]$ with $w_i = \frac{x+y}{2} + \frac{y-x}{2N}i$. It follows from the choice of N that for every $-N+1 \le i \le N-1$, $\|w_{i+1} - w_{i-1}\| < \delta/2$. Hence, $B(w_i, 2\|w_{i+1}-w_{i-1}\|) \subset B(1+\delta)$ and by Remark 2.4(i),

$$\Big\|f(w_i) - \frac{f(w_{i-1})+f(w_{i+1})}{2}\Big\| \le 6\varepsilon\|w_{i+1}-w_{i-1}\| = 6\varepsilon\frac{\|x-y\|}{N}. \tag{2.10}$$

Thus we have

$$\begin{aligned}\Big\|f\Big(\frac{x+y}{2}\Big)-\frac{f(x)+f(y)}{2}\Big\| &= \Big\|f(w_0)-\frac{f(w_{-N})+f(w_N)}{2}\Big\| \\ &\le \sum_{|i|\le N-1}(N-|i|)\|f(w_i)-\frac{f(w_{i-1})+f(w_{i+1})}{2}\| \\ &\le 6\varepsilon\|x-y\|\sum_{|i|\le N-1}\frac{(N-|i|)}{N}=6\varepsilon\|x-y\|N \\ &\le 6\varepsilon\|x-y\|(2\frac{\|x-y\|}{\delta}+1)<24\varepsilon\frac{\|x-y\|^2}{\delta} \\ &\le \frac{48}{\delta}\varepsilon\|x-y\|,\end{aligned}$$

where the first inequality follows from (2.9), the second follows from (2.10), the third from the choice of N, the fourth from the assumption on δ, and the last from $\|x-y\|\le 2$. ∎

Remark 2.6: (i) Note that by the same proof, we have:

Let $A\subset X$ be convex with $\operatorname{diam} A\le 2$. *Let $0<\delta\le 4$, and let $f\colon B(A,\delta)\to Y$ be ε-quasi-isometric. Then f is homogeneously $48\varepsilon/\delta$-AML on A.*

(ii) Matouškova [M] gave a direct proof of Lemma 2.5 when X and Y are Hilbert spaces. She used geometric properties of the Euclidean norm.

PROBLEM: *We do not know whether (for $\delta=0$) an ε-quasi-isometric map $f\colon B_X\to Y$ is necessarily homogeneously $k(\varepsilon)$-AML on B_X with $k(\varepsilon)\searrow 0$ as $\varepsilon\searrow 0$.*

This would follow, for example, from an affirmative answer to the question:

> *Do there exist a constant $\delta>0$ (independent of ε) and a positive function $\gamma(\varepsilon)$ such that $\gamma(\varepsilon)\searrow 0$ as $\varepsilon\searrow 0$ and every ε-quasi-isometry $f\colon B_X\to Y$ can be extended as a $\gamma(\varepsilon)$-quasi-isometry to $B_X(1+\delta)$?*

Actually, a weaker statement would suffice, namely, an affirmative answer to the question:

> *Given normed spaces X and Y, do there exist positive functions $\gamma(\varepsilon)$ and $\delta(\varepsilon)$ such that $\gamma(\varepsilon)/\delta(\varepsilon)\searrow 0$ as $\varepsilon\searrow 0$ and every ε-quasi-isometry $f\colon B_X\to Y$ can be extended as a $\gamma(\varepsilon)$-quasi-isometry to $B_X(1+\delta(\varepsilon))$?*

Note that when $f\colon B_X \to Y$ is an ε-quasi-isometry, then for all $x, y \in B_X$

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \le c\varepsilon,$$

where c is an absolute constant.

Indeed, by the Main Lemma f is homogeneously 6ε-AML on any ball $\bar{B}_X(w,t)$ for which $B(w,5t) \subset B_X$. Let $u, v \in B_X$ ($u \neq v$) and denote $E = \operatorname{span}\{u,v\}$ and $f_E = f|_{B_E}$. Then f_E is homogeneously 6ε-AML on $\bar{B}_X(w,t) \cap E$ whenever $B(w,5t) \subset B_X$. By [V1, Proposition 3.2] applied to f_E on $\bar{B}_E(1/5)$, there is a linear operator $F\colon E \to Y$ such that

$$\|f_E(x) - f(0) - Fx\| \le C\varepsilon$$

for some absolute constant C and for every $x \in \bar{B}_E(1/5)$. Certainly, the map $g = f_E - f(0) - F$ is homogeneously 6ε-AML on any ball $\bar{B}_E(w,t)$ for which $B_E(w,5t) \subset B_E$; and it is Lipschitzian on B_E and bounded on $\bar{B}_E(1/5)$ by $C\varepsilon$. By scaling and by Lemma 3.1 below, we obtain an absolute constant K such that $\|g(x)\| \le K\varepsilon$ for every $x \in B_E$. Hence

$$\left\| f\left(\frac{u+v}{2}\right) - \frac{f(u)+f(v)}{2} \right\| = \left\| g\left(\frac{u+v}{2}\right) - \frac{g(u)+g(v)}{2} \right\| \le 2K\varepsilon. \qquad \blacksquare$$

We finish the section by observing that Lemmas 2.1, 2.5 and [V1, Lemmas 2.5, 2.7, 2.8] imply that in a sense an ε-quasi-isometry of a ball is "nearly affine" on proper sub-balls.

COROLLARY 2.7: *Let $f\colon B_X \to Y$ be ε-quasi-isometric. Put, for $0 < \rho < 1$,*

$$\psi(\rho) = \begin{cases} 6, & 0 < \rho \le 1/5, \\ \frac{48}{1/\rho - 1}, & 1/5 < \rho < 1. \end{cases}$$

Then for each $0 < \rho < 1$, f is homogeneously $\psi(\rho)\varepsilon$-AML on $\bar{B}(\rho)$ and

(i)

$$\left\| f\left(\sum_{i=1}^{m} \lambda_i x_i\right) - \sum_{i=1}^{m} \lambda_i f(x_i) \right\| \le 2(\log_2(m-1)+2)\psi(\rho)\rho\varepsilon$$

for every integer $m \ge 2$, $\{x_i\}_{i=1}^m \subset \bar{B}(\rho)$ and $\lambda_i \ge 0$ with $\sum_{i=1}^m \lambda_i = 1$.

(ii) *There is an absolute constant k such that if X has non-trivial type p, then*

$$\left\| f\left(\sum \lambda_i x_i\right) - \sum \lambda_i f(x_i) \right\| \le \frac{k}{p-1}(1 + |\log_2(p-1)| + \log_2 T_p(X))\psi(\rho)\rho\varepsilon$$

for every $\{x_i\} \subset \bar{B}(\rho)$ and $\lambda_i \ge 0$ with $\sum \lambda_i = 1$.

(iii) *If* $X = L_p(\mu)$ *for some* $2 \le p \le \infty$, *then*

$$\left\| f\left(\sum \lambda_i x_i\right) - \sum \lambda_i f(x_i)\right\| \le 12816\psi(\rho)\rho\varepsilon$$

for every $\{x_i\} \subset \bar{B}(\rho)$ *and* $\lambda_i \ge 0$ *with* $\sum \lambda_i = 1$.

(iv) *Let* $1 < p \le \infty$ *and* $M > 0$. *Let a sequence* $\{u_i\}_{i\in I} \subset \bar{B}_X$ *satisfy*

$$\left\| \sum_{j\in J} \theta_j u_j \right\| \le M|J|^{1/p}$$

for every finite set $J \subseteq I$ *with* $|J| \ge 2$ *and for every* $\theta_j \in \{-1, 1\}$. *Let* $A = \overline{\mathrm{co}}(\{\rho u_i\}_{i\in I})$. *Then*

$$\left\| f\left(\sum \lambda_i x_i\right) - \sum \lambda_i f(x_i)\right\| \le \frac{4M}{2^{(p-1)/p} - 1}\psi(\rho)\rho\varepsilon$$

for every $\{x_i\} \subset A$ *and* $\lambda_i \ge 0$ *with* $\sum \lambda_i = 1$.

3. Affine approximation of quasi-isometries

The following technical lemma gives global bounds for a locally homogeneous AML function in terms of its bound on a fixed sub-ball.

LEMMA 3.1: *Let* X *and* Y *be normed spaces, and let* Ω *be a convex subset of* X *which contains the ball* $B(r)$ *with* $r > 1$. *Let* $g\colon \Omega \to Y$ *be a continuous function which is homogeneously* ε*-AML on any ball* $\bar{B}(w, t)$ *for which* $B(w, rt) \subset \Omega$. *Then*

$$\|g(x)\| \le (\sup_{y\in B} \|g(y)\| + 4\varepsilon\|x\|)(2\|x\| + 1) \tag{3.1}$$

for all $x \in \Omega$.

Proof: Putting $\lambda = \|x\|/(2\|x\| + 1)$ for every $x \in \Omega$, we shall show that

$$\left\|(1-\lambda)g\left(\frac{x}{\|x\|+1}\right) - \lambda g(0) - (1-2\lambda)g(x)\right\| \le 4\varepsilon\|x\|. \tag{3.2}$$

This implies (3.1) because $1/(1-2\lambda) = 2\|x\| + 1$.

We shall use the following fact (see [V1, Lemma 2.5]):

If A *is a convex set on which* g *is homogeneously* ε*-AML, then*

$$\|g(\mu u + (1-\mu)v) - \mu g(u) - (1-\mu)g(v)\| \le 2\varepsilon\|u - v\| \tag{3.3}$$

for all $u, v \in A$ *and* $0 \le \mu \le 1$.

The proof of (3.2) will be done in two steps.

Assume first that $x \in \bar{B}$. By the assumptions g is homogeneously ε-AML on $\bar{B}$, hence (3.3) gives

$$\begin{aligned}\Big\|(1-\lambda)g\Big(\frac{x}{\|x\|+1}\Big)&-\lambda g(0)-(1-2\lambda)g(x)\Big\|\\ &=(1-\lambda)\Big\|g\Big(\frac{x}{\|x\|+1}\Big)-\frac{\lambda}{1-\lambda}g(0)-\frac{1-2\lambda}{1-\lambda}g(x)\Big\|\\ &\le 2(1-\lambda)\varepsilon\|x\|\le 2\varepsilon\|x\|.\end{aligned}$$

In the second step assume that $x \in \Omega \setminus \bar{B}$. Put $q = \|x\|/(\|x\|+1)$ and $x_j = (1-q^j)x$ for $j = 0, 1, \ldots$. Then

$$\|x_j - x_{j+1}\| < \|x_j - x_{j-1}\| = |1 - 1/q|q^j\|x\| = q^j$$

by the definition of q. Thus $x_{j-1}, x_{j+1} \in \bar{B}(x_j, q^j)$. Since Ω is convex and contains $B(r)$ and x, $B(x_j, rq^j) \subset \Omega$. Then by the assumptions, g is homogeneously ε-AML on $\bar{B}(x_j, q^j)$. Now observing that $x_j = \lambda x_{j-1} + (1-\lambda)x_{j+1}$ we obtain by (3.3)

$$\|g(x_j) - \lambda g(x_{j-1}) - (1-\lambda)g(x_{j+1})\| \le 2\varepsilon\|x_{j-1} - x_{j+1}\| < 4\varepsilon q^j.$$

Hence for every $i \ge 2$,

$$\begin{aligned}&\|(1-\lambda)g(x_1) - \lambda g(0) + \lambda g(x_i) - (1-\lambda)g(x_{i+1})\|\\ &\qquad\le \sum_{j=1}^{i}\|g(x_j) - \lambda g(x_{j-1}) - (1-\lambda)g(x_{j+1})\| \le 4\varepsilon q/(1-q) = 4\varepsilon\|x\|.\end{aligned}$$

Letting $i \longrightarrow \infty$ completes the proof. ∎

PROPOSITION 3.2: *Let* $\dim X = \dim Y = n$ *and* Ω *be a convex domain of* X *which contains a ball* $B(z,r)$ *and is contained in a concentric ball* $B(z,R)$. *Let* $f\colon \Omega \to Y$ *be an* ε*-quasi-isometry. Then there is an absolute constant* k *such that*

(i) *For every* $1 < p \le 2$, *there is a linear operator* $F\colon X \to Y$ *such that*

$$\text{(3.4)}\quad \begin{aligned}&\|f(x) - f(z) - F(x-z)\|\\ &\quad\le \frac{k}{p-1}(1 + |\ln(p-1)| + \ln T_p(X))\min\{d(l_1^n, X), d(l_\infty^n, Y)\}\varepsilon R^2/r\end{aligned}$$

for every $x \in \Omega$ *and*

$$\text{(3.5)}\quad \Big|\frac{\|Fx\|}{\|x\|} - 1\Big| \le \frac{k}{p-1}(1 + |\ln(p-1)| + \ln T_p(X))\min\{d(l_1^n, X), d(l_\infty^n, Y)\}\varepsilon$$

for every $x \neq 0$.

Obviously, if

$$\frac{k}{p-1}(1+|\ln(p-1)|+\ln T_p(X))\min\{d(l_1^n,X),d(l_\infty^n,Y)\}\varepsilon<1,$$

then F *is an onto isomorphism.*

(ii) *If* $X = l_p^n$ $(2 \le p \le \infty)$, *then there is a linear operator* $F\colon l_p^n \to Y$ *such that*

$$\|f(x)-f(z)-F(x-z)\| \le k\min\{\sqrt{n},d(l_\infty^n,Y)\}\varepsilon R^2/r \tag{3.6}$$

for every $x \in \Omega$ *and*

$$\left|\frac{\|Fx\|}{\|x\|}-1\right| \le k\min\{\sqrt{n},d(l_\infty^n,Y)\}\varepsilon \tag{3.7}$$

for every $x \neq 0$.

If $k\min\{\sqrt{n},d(l_\infty^n,Y)\}\varepsilon < 1$, *then* F *is an onto isomorphism.*

(iii) *If* $X = L_\infty(\mu)$ *and* Y *is isomorphic to* $l_\infty(\Gamma)$ *for some set* Γ *(the spaces are not necessarily finite-dimensional), then we have*

$$\|f(x)-f(z)-F(x-z)\| \le kd(l_\infty(\Gamma),Y)\varepsilon R^2/r \tag{3.8}$$

for every $x \in \Omega$ *and*

$$\left|\frac{\|Fx\|}{\|x\|}-1\right| \le kd(l_\infty(\Gamma),Y)\varepsilon \tag{3.9}$$

for every $x \neq 0$.

If $kd(l_\infty(\Gamma),Y)\varepsilon < 1$, *then* F *is an onto isomorphism.*

(iv) *Let* $1 < p \le \infty$ *and* $M > 0$. *Suppose that* X *has a normalized basis* $\{e_i\}_{i=1}^n$ *satisfying*

$$\left\|\sum_{j\in J}\theta_j e_j\right\| \le M|J|^{1/p}$$

for every set $J \subseteq \{1,\dots,n\}$ *with* $|J| \ge 2$ *and for every* $\theta_j \in \{-1,1\}$. *Put* $\beta = \min_{\sum|t_i|=1}\|\sum t_i e_i\|$. *Then there is a linear operator* $F\colon X \to Y$ *such that*

$$\|f(x)-f(z)-F(x-z)\| \le \frac{kM}{(p-1)\beta}\varepsilon R^2/r \tag{3.10}$$

for every $x \in \Omega$ *and*

$$\left|\frac{\|Fx\|}{\|x\|}-1\right| \le \frac{kM}{(p-1)\beta}\varepsilon \tag{3.11}$$

for every $x \neq 0$.

If $\frac{kM}{(p-1)\beta}\varepsilon < 1$, *then* F *is an onto isomorphism.*

Proof: We only prove (iii). The proofs of the remaining assertions follow the same path using [V1, Propositions 3.2, 3.4, 3.9]. We omit the details.

We shall use the next claim which follows from [V2, Proposition 1.3.6(iv)] and is, in fact, an immediate corollary of [BK, Theorem 3.9(c)].

There is an absolute constant K *so that if* Y *is isomorphic to* $l_\infty(\Gamma)$ *for some set* Γ, $2 \leq p \leq \infty$ *and* $f\colon \bar{B}_{L_p(\mu)} \to Y$ *is a continuous homogeneously* ε*-AML function, then there is an affine function* $h\colon L_p(\mu) \to Y$ *such that*

$$\|f(x) - h(x)\| \leq Kd(l_\infty(\Gamma), Y)\varepsilon$$

for all $x \in \bar{B}_{L_p(\mu)}$.

We can assume that $\varepsilon < 1$, because otherwise we put $F = 0$ and then

$$\|f(x) - f(z)\| \leq (1+\varepsilon)\|x - z\| \leq (1 + 1/\varepsilon)\varepsilon R \leq 2\varepsilon R^2/r.$$

Lemmas 2.3(i) and 2.1 imply that f is ε-rigid and that it is homogeneously 6ε-AML on $\bar{B}(w,t)$, provided $B(w, 5t) \subset \Omega$. By translation and scaling, we can assume that $z = 0$, $f(0) = 0$ and $r = 5$. Thus $B(z,r) = B(5)$.

Since f is homogeneously 6ε-AML on $\bar{B}$, the claim above gives an absolute constant K and a linear operator $F\colon X \to Y$ so that

$$\|f(x) - Fx\| \leq Kd(l_\infty(\Gamma), Y)\varepsilon \tag{3.12}$$

for every $x \in \bar{B}$. This with the ε-rigidity of f imply for every x with $\|x\| = 1$,

$$|\|Fx\| - 1| \leq |\|Fx\| - \|f(x)\|| + |\|f(x)\| - 1| \leq Kd(l_\infty(\Gamma), Y)\varepsilon + \varepsilon.$$

Thus, F satisfies (3.9).

Let $x \in \Omega$. The function $g = f - F$ is homogeneously 6ε-AML on $\bar{B}(w,t)$, whenever $B(w, 5t) \subset \Omega$, and Lipschitzian on Ω. By Lemma 3.1 (with $r = 5$) and by (3.12),

$$\|g(x)\| \leq (Kd(l_\infty(\Gamma), Y) + 24\|x\|)(2\|x\| + 1)\varepsilon \leq kd(l_\infty(\Gamma), Y)R^2\varepsilon,$$

which completes the proof, since scaling yields the $1/r$ factor in the right-hand side of (3.8). ∎

By Lemma 2.3, given an ε-quasi-isometry $f\colon B_X(z,r) \to Y$, its inverse map f^{-1} on $B_Y(f(z), r/(1+\varepsilon)^3)$ is well-defined, and it is also ε-quasi-isometric. So it can also be approximated by a linear operator (from Y to X) as above. This simple observation enables us to improve Proposition 3.2(i) for small ε's as follows.

COROLLARY 3.3: *Let the assumption of Proposition* 3.2(i) *hold, and put, for every* $1 < p \leq 2$,

$$C_p(X,Y) = \frac{k}{p-1}(1 + |\ln(p-1)| + \ln\max\{T_p(X), T_p(Y)\}) \cdot \min\{d(l_1^n, X), d(l_\infty^n, X), d(l_1^n, Y), d(l_\infty^n, Y)\},$$

where k *is an absolute constant from Proposition* 3.2. *Then there is an absolute constant* k_1 *such that if* $C_p(X,Y)\varepsilon \leq 1/2$ *for some* p, *then there is an onto isomorphism* $F\colon X \to Y$ *so that*

$$\|f(x) - f(z) - F(x-z)\| \leq k_1 C_p(X,Y)\varepsilon R^2/r$$

for every $x \in \Omega$ *and*

$$\|F\|\|F^{-1}\| \leq 1 + 4C_p(X,Y)\varepsilon.$$

Proof: Put $a = C_p(X,Y)\varepsilon$.

If

$$a \geq \frac{k}{p-1}(1 + |\ln(p-1)| + \ln T_p(X))\min\{d(l_1^n, X), d(l_\infty^n, Y)\}\varepsilon,$$

the statement is true by Proposition 3.2(i).

If this inequality does not hold, then

$$a \geq \frac{k}{p-1}(1 + |\ln(p-1)| + \ln T_p(Y))\min\{d(l_\infty^n, X), d(l_1^n, Y)\}\varepsilon.$$

Assume, as we may, that $z = 0$, $f(0) = 0$ and $r = (1+\varepsilon)^4$. By Lemma 2.3(ii), $f(B_X) \subseteq B_Y(1+\varepsilon) \subseteq f(\Omega)$. As f^{-1} is an ε-quasi-isometry on $B_Y(1+\varepsilon)$, Proposition 3.2(i) gives a linear operator $G\colon Y \to X$ such that

$$\|f^{-1}(y) - Gy\| \leq a(1+\varepsilon) \quad \text{for every } y \in B_Y((1+\varepsilon)) \tag{3.13}$$

and

$$\left|\frac{\|Gy\|}{\|y\|} - 1\right| \leq a \quad \text{for every } y \neq 0. \tag{3.14}$$

Since $a \leq 1/2$, the inverse operator $F = G^{-1}$ exists and

$$\|F\|\|F^{-1}\| \leq (1+a)/(1-a) \leq 1 + 4a.$$

Let $x \in B_X$. Put $y = f(x)$. It follows from (3.13) and (3.14) that

$$\|f(x) - Fx\| \leq \|F\|\|Gy - f^{-1}(y)\| \leq a(1+\varepsilon)/(1-a) \leq 2a(1+\varepsilon). \tag{3.15}$$

Now one can complete the proof by the same argument as in the proof of Proposition 3.2 using (3.15) instead of (3.12). ∎

The following statement was obtained in [V2, Corollary 1.3.11], and we quote it without proof. (The proof is a modification of Kalton's [K], who established it, in fact, for $X = Y = E = l_2^n$ as a tool in his proof that the estimate $k(\log n + 1)\varepsilon R^2/r$ in John's Theorem 1.1 is sharp.)

There is an absolute constant C with the following property: Let X and Y be n-dimensional real Banach spaces, and let $f\colon \bar{B}_X \to Y$ be a bounded function with $f(0) = 0$ and

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \le K$$

for all $x, y \in \bar{B}_X$. Then for every n-dimensional real Banach space E there exists a linear operator F with

$$\|f(x) - Fx\| \le CT_2(E)^2 d(E,X) d(E^*,Y)(\ln n + 1)K$$

for every $x \in \bar{B}_X$.

Now similarly to Corollary 3.3, by ensuring the invertibility of the approximating linear operator from Y to X, this statement and Lemmas 2.1 and 3.1 imply

PROPOSITION 3.4: *There are absolute constants k_1 and k_2 with the following property: Let X, Y and E be n-dimensional real Banach spaces ($n \ge 2$). Let a convex domain Ω be such that $\bar{B}(z,r) \subseteq \Omega \subseteq B(z,R) \subset X$, and let $f\colon \Omega \to Y$ be an ε-quasi-isometry. Put*

$$C(X,Y;E) = T_2(E)^2 \min\{d(E,X)d(E^*,Y), d(E,Y)d(E^*,X)\} \ln n.$$

If $C(X,Y;E)\varepsilon \le k_1$, then there is an onto isomorphism $F\colon X \to Y$ so that

$$\|f(x) - f(z) - F(x-z)\| \le k_2 C(X,Y;E)\varepsilon R^2/r$$

for every $x \in \Omega$ and

$$\left| \frac{\|Fx\|}{\|x\|} - 1 \right| \le k_2 C(X,Y;E)\varepsilon$$

for every $x \ne 0$.

We can combine Proposition 3.4 with Corollary 3.3 as follows:

Let E be an n-dimensional real Banach space and put

$$\iota(E) = \sup\{\min\{d(l_1^n,X), d(l_\infty^n,X), T_2(E)^2 d(E,X)d(E^*,X)\} : X \text{ is a real } n\text{-dimensional space}\}.$$

Define for every natural n

$$\iota_n = \inf\{\iota(E) \colon E \text{ is a real } n\text{-dimensional space}\}.$$

COROLLARY 3.5: *There are absolute constants k_1 and k_2 with the following property: Let X and Y be n-dimensional real Banach spaces ($n \geq 2$), and assume that $\ln(n)\iota_n\varepsilon \leq k_1$. Let a convex domain Ω be such that $\bar{B}(z,r) \subseteq \Omega \subseteq B(z,R) \subset X$, and let $f\colon \Omega \to Y$ be an ε-quasi-isometry. Then there is an onto isomorphism $F\colon X \to Y$ so that*

$$\|f(x) - f(z) - F(x-z)\| \leq k_2 \ln(n)\iota_n\varepsilon R^2/r$$

for every $x \in \Omega$ and

$$\left|\frac{\|Fx\|}{\|x\|} - 1\right| \leq k_2 \ln(n)\iota_n\varepsilon$$

for every $x \neq 0$.

Proof: Put $a = \min\{d(l_1^n, X), d(l_\infty^n, X), d(l_1^n, Y), d(l_\infty^n, Y)\}$.

If $a \leq \iota_n$, then the statement follows from Corollary 3.3, because $T_2(E) \leq \sqrt{n}$ for every n-dimensional real Banach space E.

Otherwise, choose for every $s \in (0, a - \iota_n)$ an n-dimensional real Banach space E_s such that $\iota(E_s) < \iota_n + s < a$. Then by the definition of $\iota(E_s)$,

$$T_2(E_s)^2 d(E_s, X) d(E_s^*, X) \leq \iota(E_s) < \iota_n + s$$

and

$$T_2(E_s)^2 d(E_s, Y) d(E_s^*, Y) < \iota_n + s.$$

Now the statement follows from Proposition 3.4. ∎

Propositions 3.2 and 3.4 and some known results in the linear theory imply that ε-quasi-isometries in "nice" spaces can be approximated by linear isometries. (Corollary 3.6(iv) below is due to Kalton [K].)

COROLLARY 3.6: *There are absolute constants K_1, K_2, K_3, K_4 and K_5 and a function $\eta(p,s)$ with $\eta(p,s) \searrow 0$ as $s \searrow 0$ so that, whenever a convex domain Ω is such that $\bar{B}(z,r) \subseteq \Omega \subseteq B(z,R) \subset l_p^n$ ($1 \leq p \leq \infty$, $n \geq 2$) and $f\colon \Omega \to l_p^n$ is an ε-quasi-isometry, one has:*

(i) *If $p = 1$ and $\ln(n)\varepsilon \leq K_1$, then there is a linear isometry W of l_1^n such that*

$$\|f(x) - f(z) - W(x-z)\| \leq K_2 \ln(n)\varepsilon R^2/r$$

for every $x \in \Omega$.

(ii) *If* $1 < p \leq 4/3$ *and* $n^{(p-1)/p}\varepsilon$ *is sufficiently small, then there is a linear isometry* W *of* l_p^n *such that*

$$\|f(x) - f(z) - W(x - z)\| \leq \eta(p, n^{(p-1)/p}\varepsilon)R^2/r$$

for every $x \in \Omega$.

(iii) *If* $p \in (4/3, 2) \cup (2, 4)$ *and* $\ln(n)n^{|1/p-1/2|}\varepsilon$ *is sufficiently small, then there is a linear isometry* W *of* l_p^n *such that*

$$\|f(x) - f(z) - W(x - z)\| \leq \eta(p, \ln(n)n^{|1/p-1/2|}\varepsilon)R^2/r$$

for every $x \in \Omega$.

(iv) *If* $p = 2$*, then there is a linear isometry* W *of* l_2^n *such that*

$$\|f(x) - f(z) - W(x - z)\| \leq K_3 \ln(n)\varepsilon R^2/r$$

for every $x \in \Omega$.

(v) *If* $4 \leq p < \infty$ *and* $n^{1/p}\varepsilon$ *is sufficiently small, then there is a linear isometry* W *of* l_p^n *such that*

$$\|f(x) - f(z) - W(x - z)\| \leq \eta(p, n^{1/p}\varepsilon)R^2/r$$

for every $x \in \Omega$.

(vi) *If the source space is* $L_\infty(\mu)$ *and the target space is* $l_\infty(\Gamma)$ *for some set* Γ *(the spaces are not necessarily finite-dimensional) and if* $\varepsilon \leq K_4$*, then there is a linear isometry* W *of* $L_\infty(\mu)$ *onto* $l_\infty(\Gamma)$ *such that*

$$\|f(x) - f(z) - W(x - z)\| \leq K_5\varepsilon R^2/r$$

for every $x \in \Omega$.

This improves and generalizes Theorem 1.1.

Proof: Assume again that $z = 0$, $f(0) = 0$ and $r = 1$.

(i) Since $T_2(X) \leq \sqrt{n}$ for every n-dimensional space X, it follows from Proposition 3.2(i) (this time with the type $p = 2$) that there are an absolute constant k and an isomorphism $F\colon l_1^n \to l_1^n$ such that

$$\|f(x) - Fx\| \leq k \ln(n)\varepsilon R^2$$

for every $x \in \Omega$ and

$$\left|\frac{\|Fx\|}{\|x\|} - 1\right| \leq k \ln(n)\varepsilon$$

for every $x \neq 0$. Thus by Godefroy, Kalton and Li [GKL, Theorem II.7], for example, if $k\ln(n)\varepsilon \leq 1/26$, then there is a linear isometry W of l_1^n so that

$$\|(1-k\ln(n)\varepsilon)^{-1}F - W\| \leq 26(1-k\ln(n)\varepsilon)^{-1}k\ln(n)\varepsilon.$$

Then W satisfies the conclusion of the statement.

(ii) Denote by $\{e_i\}_{i=1}^n$ the standard unit vector basis of l_p^n. Then Proposition 3.2(iv) holds with $M=1$ and $\beta = n^{-(p-1)/p}$. Now the proof of the claim is completed in the same path as in (i) with use of a theorem of Alspach [Al] instead of the Godefroy–Kalton–Li theorem.

(iii) follows from Proposition 3.4 (with $E=(l_p^n)^*$ for $p<2$ and $E=l_p^n$ for $p>2$) and Alspach's theorem, since for every $2 \leq q < \infty$, $T_2(l_q^n) \leq c\sqrt{q}$ and $d(l_q^n,(l_q^n)^*) \leq Cn^{1/2-1/q}$, where c and C are absolute constants (see [T, p. 15] and [T, Proposition 37.6 on p. 280]).

(iv) This case is handled easily by applying Proposition 3.4 (with $E=l_2^n$) and using then the polar decomposition.

(v) follows from Proposition 3.2(ii), $d(l_\infty^n, l_p^n) = n^{1/p}$ for $2<p<\infty$ and Alspach's theorem as above.

(vi) Recall the well-known fact that any space $L_\infty(\mu)$ is linear isometric to a $C(S)$ space for some compact Hausdorff S. Now the claim follows from Proposition 3.2(iii) and the next result due to Amir [Am] and Cambern [C].

Let K and S be compact Hausdorff spaces. If there is a linear operator T of $C(K)$ onto $C(S)$ such that $\|f\| \leq \|Tf\| \leq (1+\varepsilon)\|f\|$ for some $0<\varepsilon<1$, then there is a linear isometry W of $C(K)$ onto $C(S)$ such that $\|T-W\| \leq 3\varepsilon$. ∎

Remark 3.7: (i) As has been shown by Matoušková [M] and Kalton [K], the estimate in Corollary 3.6(iv) is sharp.

(ii) In the simple case when $X=Y=\mathbf{R}$, an ε-quasi-isometry

$$f\colon (z-r, z+r) \longrightarrow \mathbf{R}$$

is ε-rigid in its domain. Hence, the linear isometry

$$Fx = x\,\mathrm{sgn}(\lim_{t\to r}(f(z+t)-f(z-t)))$$

satisfies

$$\|f(x)-f(z)-F(x-z)\| \leq \varepsilon\|x\|.$$

4. Injectivity of quasi-isometries

Following tradition, we use here the notation of (m, M)-quasi-isometries instead of ε-quasi-isometries, and we set $\mu = M/m$. (Note that if f is an (m, M)-quasi-isometry, then $f/\sqrt{Mm}$ is $(\sqrt{M/m}-1)$-quasi-isometric, and $\mu = (1+\varepsilon)^2$ for an ε-quasi-isometry.) Recall first some definitions (cf. [G2]).

For a given connected open subset U of a Banach space X, we define $\mu_0(U)$ to be the infimum of all μ for which there exists a noninjective $(m, \mu m)$-quasi-isometry from U into some Banach space Y.

We say that $U \subset X$ is (r, R)-**convex** if it is open and convex and $B(z, r) \subset U \subset B(z, R)$ for some $z \in X$. We also define for $0 < \tau \leq 1$

$$\mu_0(\tau) = \inf\{\mu_0(U) \colon U \text{ is } (r, R)\text{-convex}, r/R \geq \tau\}$$

and $\mu_0 = \mu_0(1)$. (Note that $\mu_0(\tau)$ is unchanged if we only take $r/R = \tau$ in its definition, since if U contains $B(z, r)$ then it contains $B(z, r')$ with $r' < r$.)

The following concept was introduced by Martio and Sarvas [MS], but the formulation given here is taken from Gevirtz [G2]. We say that an open subset $U \subset X$ is an (a, b)-**uniform domain** if any two points x, y of U may be joined by a curve $C \subset U$ with the following properties:

(1) C has finite length $L \leq a\|x - y\|$.

(2) If $\gamma\colon [0, L] \longrightarrow X$ is the arc length parameterization of C, then $B(\gamma(t),\ b\min\{t, L-t\}) \subset U$ for all $t \in [0, L]$.

In this section we shall establish that the function $\mu_0(\tau)$ behaves linearly near zero. We shall obtain also lower bounds for $\mu_0(U)$, where U is an (a, b)-uniform domain. We shall use some arguments of Gevirtz [G2] and our Main Lemma. It is evident that the function $\mu_0(\tau)$ is non-decreasing and that for every bounded domain U, $\mu_0(U) \leq \mu_0$. Gevirtz [G1] showed that $\mu_0 \geq 1.114\ldots$ (this is the best known estimate).

Let $e = (1, 0) \in l_2^2$. Then the map $f\colon B(e, 1) \to l_2^2$, given in polar coordinates by $f(r, \theta) = f(r, \alpha\theta)$, is $(1, \alpha)$-quasi-isometric, but is not injective when $\alpha > 2$. It follows that $\mu_0 \leq 2$, and we shall thus restrict ourselves, mainly, to $(m, \mu m)$-quasi-isometries with $\mu \leq 2$.

The next lemma is just a reformulation of Remark 2.4(i) to the language of (m, M)-quasi-isometries.

LEMMA 4.1: *Let X and Y be Banach spaces. Let $x, y \in X$, and let*

$$f\colon B((x+y)/2, 2\|x - y\|) \longrightarrow Y$$

be (m, M)-quasi-isometric. Put $\mu = M/m$. Then

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leq 6\left(1 - \frac{1}{\sqrt{\mu}}\right) M \|x-y\|.$$

Using Lemma 4.1 and arguments of Gevirtz [G2] one can obtain

PROPOSITION 4.2: *Let X and Y be Banach spaces, and let $0 < m \leq M$ with $\mu = M/m \leq 2$.*

(i) *Let $x, y \in X$, $\delta > 0$, and let $f\colon B([x,y],\delta) \longrightarrow Y$ be (m, M)-quasi-isometric. Then*

$$\|f(x) - f(y)\| \geq m(1 - 48(\mu - \sqrt{\mu})\|x-y\|/\delta)\|x-y\|.$$

(ii) *Let $U \subset X$ be an (r, R)-convex domain, and let $f\colon U \to Y$ be (m, M)-quasi-isometric. If*

$$384(\mu - \sqrt{\mu})\mu < r/R, \tag{4.1}$$

then f is injective.

(iii) *Let $x, y \in X$, $\delta > 0$, and let C be a curve of length L joining x to y. Let $f\colon B(C,\delta) \longrightarrow Y$ be (m, M)-quasi-isometric. Then*

$$\|f(x) - f(y)\| \geq m(\|x-y\| - 360(\mu - \sqrt{\mu})L^2/\delta).$$

(iv) *Let $U \subset X$ be an (a, b)-uniform domain, and let $f\colon U \to Y$ be (m, M)-quasi-isometric. If*

$$\mu + 6480(\mu - \sqrt{\mu})a^2/b < 2, \tag{4.2}$$

then f is injective.

The proofs of these statements are exactly the same as the proofs by Gevirtz of Lemmas 10, 11 and Theorems 1, 2, 3 and 4 in [G2, pp. 313–317]; the only distinction is the use of Lemma 4.1 instead of Proposition 2 from [G2, p. 313]. We refer the reader to this article for details.

COROLLARY 4.3: *Denote the unique solution of the equation*

$$384(s - \sqrt{s})s = \tau$$

by s_τ. Then

(i) $\mu_0(\tau) \geq s_\tau$.

(ii) $\mu_0(\tau) \geq 1 + k\tau$, *where* $k = (\sqrt{s_1} + 1)/384 s_1^{1.5} = s_1 - 1 \approx 0.0052$.

Proof: Since $(t - \sqrt{t})t$ increases for $t \geq 1$, every μ, such that $1 \leq \mu < s_\tau$, satisfies (4.1) for $r \geq \tau R$. Hence, (i) follows by the definition of $\mu_0(\tau)$ and Proposition 4.2(ii).

Similarly, $(\sqrt{t} + 1)/t^{1.5}$ decreases for $t > 0$, hence

$$k\tau \leq \frac{\sqrt{s_\tau} + 1}{384 s_\tau^{1.5}} \tau = s_\tau - 1 \quad \text{for } 0 < \tau \leq 1.$$

Thus $s_\tau \geq 1 + k\tau$, so (ii) follows from (i). ∎

Remark 4.4: Corollary 4.3(ii) answers a question of Gevirtz: In [G2, Corollary] he showed that $\mu_0(\tau) \geq 1 + k_1 \tau^{k_2}$ with $k_1 \approx 1.7(10)^{-19}$ and $k_2 \approx 8.22$, and posed the question (see [G2, Remark 3]) whether it is possible to take $k_2 = 1$ with a suitable value of k_1.

That k_2 cannot be smaller than 1 follows from the next example of John [J5]:

For a given $\varepsilon > 0$, consider the mapping h of l_2^2 into itself, given by the exponential function

$$h(z) = e^{\varepsilon z}/\varepsilon$$

of a complex variable z. Direct computations show that h is $(e^{-\varepsilon}, e^{\varepsilon})$-quasi-isometric in the strip $|\operatorname{Re} z| < 1$. On the other hand,

$$h\Big(\frac{2\pi}{\varepsilon} i\Big) = \frac{1}{\varepsilon} = h(0),$$

that is, h is non-injective on $\widetilde{U} = \operatorname{co}(B \cup \{\frac{2\pi}{\varepsilon} i\})$. Therefore,

$$\mu_0\Big(\frac{\varepsilon}{2\pi}\Big) \leq \mu_0(\widetilde{U}) \leq e^{2\varepsilon} = 1 + 2\varepsilon + o(\varepsilon).$$

Note also that John [J5] obtained $\mu_0(\tau) \geq 1 + C\tau$ with some absolute constant C for the case when both spaces X and Y are Hilbertian.

Remark 4.5: It follows from the definition of (a, b)-uniform domains that $a \geq 1$. Also, it follows from the definition that for bounded domains $b \leq 1$. Indeed, suppose that points x, y lie in a bounded (a, b)-uniform domain U with $b > 1$. Let $\gamma \colon [0, L] \longrightarrow X$ be an arc with $\gamma(0) = x$ and $\gamma(L) = y$, and note that $x, y \in B(\gamma(L/2), L/2)$. By the definition, $U \supset B(\gamma(L/2), bL/2)$, and this ball contains the balls with radius $(b-1)L/2 \geq (b-1)\|x - y\|/2$ centered at x and y. But this is impossible when $\|x - y\| \approx \operatorname{diam} U$.

Note that B_{l_∞} is a bounded $(1,1)$-uniform domain, while in a Hilbert space the only $(1,1)$-uniform domain is the whole space.

Proposition 4.2(iv) with Remark 4.5 imply

COROLLARY 4.6: *Let U be an (a,b)-uniform domain. Denote the unique solution of the equation*

$$s + 6480(s - \sqrt{s})a^2/b = 2$$

by $s(a,b)$. Then

(i) $\mu_0(U) \geq s(a,b)$.

(ii) *If U is bounded, then*

$$\mu_0(U) \geq 1 + k\frac{b}{a^2}, \quad \text{where } k = \frac{2 - s(1,1)}{6480}\frac{\sqrt{s(1,1)}+1}{\sqrt{s(1,1)}} = s(1,1) - 1 \approx 0.00031.$$

Proof: (i) Since $1 < s(a,b) < 2$ and $t + 6480(t - \sqrt{t})a^2/b$ increases for all $a, b > 0$ and $t \geq 1$, then given $a, b > 0$, every μ, such that $1 \leq \mu < s(a,b)$, satisfies (4.2) with these a and b. The assertion follows by the definition of $\mu_0(U)$ and Proposition 4.2(iv).

(ii) It follows from Remark 4.5 that there is no bounded (a,b)-uniform domain with $a^2/b < 1$. Since $s(a,b) \leq s(1,1)$ for $a^2/b \geq 1$ and $(2-t)(\sqrt{t}+1)/\sqrt{t}$ decreases for $0 < t \leq 2$, then

$$k\frac{b}{a^2} \leq \frac{2 - s(a,b)}{6480}\frac{\sqrt{s(a,b)}+1}{\sqrt{s(a,b)}}\frac{b}{a^2} = s(a,b) - 1$$

for such a and b. Hence $s(a,b) \geq 1 + kb/a^2$, so (ii) follows from (i). ∎

Remark 4.7: As $\widetilde{U}$ from Remark 4.4(i) is $(2, \varepsilon/(4\pi))$-uniform (more precisely, any domain $\widetilde{V} = \mathrm{co}(B \cup V)$, where V is a small neighborhood of the point $\frac{2\pi}{\varepsilon}i$, is $(2, \varepsilon/(4\pi))$-uniform), then the linear dependence on b/a^2 in the estimate of Corollary 4.6(ii) is sharp.

ACKNOWLEDGEMENT: This paper is a part of the author's Ph.D. thesis prepared at the Technion under the supervision of Professor Yoav Benyamini. The author expresses his gratitude to him for his great guidance and help.

References

[Al] D. E. Alspach, *Small into isomorphisms on L_p spaces,* Illinois Journal of Mathematics **27** (1983), 300–314.

[Am] D. Amir, *On isomorphisms of continuous function spaces*, Israel Journal of Mathematics **3** (1965), 205–210.

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. 1, Colloquium Publication **48**, American Mathematical Society, Providence, RI, 2000.

[BK] Y. A. Brudnyi and N. J. Kalton, *Polynomial approximation on convex subsets of* $\mathbf{R}^n$, Constructive Approximation **16** (2000), 161–199.

[C] M. Cambern, *On isomorphisms with small bound*, Proceedings of the American Mathematical Society **18** (1967), 1062–1066.

[G1] J. Gevirtz, *Injectivity of quasi-isometric mappings of balls*, Proceedings of the American Mathematical Society **85** (1982), 345–349.

[G2] J. Gevirtz, *Injectivity in Banach spaces and the Mazur–Ulam theorem on isometries*, Transactions of the American Mathematical Society **274** (1982), 307–318.

[GKL] G. Godefroy, N. J. Kalton and D. Li, *Operators between subspaces and quotients of* L^1, Indiana University Mathematical Journal **49** (2000), 245–286.

[J1] F. John, *Rotation and strain*, Communications on Pure and Applied Mathematics **14** (1961), 391–413 [*Collected Papers*, Volume 2 (J. Moser, ed.), Birkhäuser, Boston, 1985, pp. 643–665].

[J2] F. John, *Quasi-isometric mappings*, Seminar dell'Instituto Nazionale di Alta Matematica (1962–1963), 462–473 [*Collected Papers*, Volume 2 (J. Moser, ed.), Birkhäuser, Boston, 1985, pp. 678–689].

[J3] F. John, *On quasi-isometric mappings. I*, Communications on Pure and Applied Mathematics **21** (1968), 77–110 [*Collected Papers*, Volume 2 (J. Moser, ed.), Birkhäuser, Boston, 1985, pp. 568–601].

[J4] F. John, *On quasi-isometric mappings. II*, Communications on Pure and Applied Mathematics **22** (1969), 265–278 [*Collected Papers*, Volume 2 (J. Moser, ed.), Birkhäuser, Boston, 1985, pp. 603–616].

[J5] F. John, *Distance changes in deformations with small strain* (Studies and essays presented to Yu-Why Chen), Math. Research Center, National Taiwan Univ., Taipei, 1970 pp. 1–15 [*Collected Papers*, Volume 2 (J. Moser, ed.), Birkhäuser, Boston, 1985, pp. 204–218].

[K] N. J. Kalton, *A remark on quasi-isometries*, Proceedings of the American Mathematical Society **131** (2003), 1225–1231.

[LS] J. Lindenstrauss and A. Szankowski, *Non-linear perturbations of isometries*, Astérisque **131** (1985), 357–371.

[MS] O. Martio and J. Sarvas, *Injectivity theorems in plane and space,* Annales Academiae Scientiarum Fennicae. Series A I. Mathematica Dissertationes **4** (1978/1979), 383–401.

[M] E. Matoušková, *Almost isometries of balls,* Journal of Functional Analysis **190** (2002), 507–525.

[MU] S. Mazur and S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés,* Comptes Rendus de l'Académie des Sciences, Paris **194** (1932), 946–948.

[N] R. Nevanlinna, *Über die Methode der sukzessiven Approximationen,* Annales Academiae Scientiarum Fennicae. Series AI **291** (1960), 10.

[T] N. Tomczak-Jaegermann, *Banach–Mazur Distance and Finite Dimensional Operator Ideals,* Pitman Monographs and Surveys in Pure and Applied Mathematics **38**, Longman Sci. Tech., Harlow, 1989.

[V1] I. A. Vestfrid, *Linear approximation of approximately linear functions,* Aequationes Mathematicae **66** (2003), 37–77.

[V2] I. A. Vestfrid, *Affine properties of almost isometries and of almost Jensen functions,* Ph.D. thesis, Technion, Haifa, 2002.